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HEAT EXPLOSION THEORY AND VIBRATIONAL HEATING OF POLYMERS

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Abstract—When testing a cylindrical polymeric sample by sinusoidal loading with a constant stress amplitude, vibrational frequencies may occur at which a "thermal explosion" leads to the destruction of the test sample. It is shown how upper and lower bounds of these critical frequencies can be derived by an application of linear methods.

NOMENCLATURE

- Bi, Biot number;
- K, parameter in the analytic expression for the heat generation;
- L, sample length;
- n, parameter in the analytic expression for the heat generation;
- r, radial distance;
- R, sample radius;
- s, secant;
- t, tangent;
- T, temperature distribution in and on the test sample;
- T_0, temperature of the surroundings of the test sample;
- W, heat generated by cyclic loading;
- y_m, maximum of the dimensionless temperature difference $y = \beta(T - T_0)$;
- z, axial distance.

Greek symbols

- α, heat-transfer coefficient;
- β, parameter in the analytic expression for the heat generation;
- δ, compounded parameter which represents the experimenter's influence on the generation of heat;
- ε, deformation, with the amplitude ε_0;
- ζ, dimensionless axial coordinate;
- κ, thermal conductivity;
- ν, frequency of the oscillating load, with $2\pi\nu = \omega$;
- ν_M, lower bound of ν;
- ν_m, upper bound of ν;
- ρ, dimensionless radial coordinate;
- σ, stress, with the amplitude σ_0.

1. INTRODUCTION

BY A LONG established procedure, mechanical properties of materials are obtained from testing samples by subjecting them to a cyclic loading. For an investigation of this type, it is of crucial importance to achieve a stationary thermal state of the sample. Such a stationary thermal state will only result if the intrinsically produced heat (the vibrational heating due to the viscous resistance of the material) is balanced by the heat transferred from the sample into its surroundings, whereas something resembling the heat explosion in exothermic chemical reactions [1, 2] occurs if such a balance cannot be established. While a stationary thermal state is easily attained for metals on account of their high thermal conductivity, polymeric materials are found to behave differently.

Let σ_0 be the stress amplitude and let ε_0 be the deformation amplitude, then the two well known testing conditions are ε_0 = constant and σ_0 = constant. It has been observed that the condition ε_0 = constant leads to a rapidly established stationary thermal state whereas for the condition σ_0 = constant, critical sample states exist beyond which the thermal explosion takes place [3].

In a recent paper [4], a method has been introduced by which a response function $\nu(y_m)$ was shown to be characteristic of the stationary thermal states of the sample. Here, ν designates the frequency of the oscillating load and y_m designates the maximum of $y = T - T_0$, where T_0 is the constant temperature of the surroundings and T is the temperature distribution in and on the test sample. In particular, an approximation method has been put forward in [4] by which continuous upper and lower bounds $\nu_m(y_m)$ and $\nu_M(y_M)$ of $\nu(y_m)$, viz.

$$\nu_M(y_m) \leqq \nu(y_m) \leqq \nu_m(y_m) \qquad (1)$$

may be derived by solving appropriately defined linear boundary value problems. Because of the strongly nonlinear dependence of the vibrational heating on the temperature, the mathematical derivation of the temperature distribution y and hence of the response function $\nu(y_m)$ is of considerable difficulty, and the introduction of a linear approximation method is therefore of an appreciable advantage. In [4], the derivation of $\nu_M(y_m)$ and $\nu_m(y_m)$ was carried out for the stationary thermal states of five plastics for which the intrinsic heat generation as a function of the temperature had been determined experimentally by [5]. Because of the test condition ε_0 = constant, no critical values of v were found. It is the aim of this paper to show that for a test condition σ_0 = constant, critical values δ_c of a parameter δ which depends on v, exist and that upper and lower bounds of δ_c may be derived by the same linear methods which were employed in [4].

2. VIBRATIONAL HEATING OF POLYMERS

Following [6–9], the generation of heat due to the viscous resistance of the polymeric material is given by a function W which is of the following form for a testing condition σ_0 = constant:

$$W = \tfrac{1}{2}\sigma_0^2 \omega^{1-n} K \exp(\beta(T - T_0)). \tag{2}$$

Here, σ_0 is the stress amplitude and $\omega = 2\pi v$ is the frequency of the oscillating axial stress σ:

$$\sigma = \sigma_0 \sin(\omega t). \tag{3}$$

The test sample is in the form of a cylinder of a radius R and of a length L. Assuming the following symmetric radial and axial boundary conditions:

$$\frac{\partial T}{\partial r} + \alpha(T - T_0) = 0 \quad \text{at } r = R,$$

$$T = 0 \quad \text{at } z = \pm\frac{L}{2} \tag{4}$$

one may introduce the following dimensionless entities [8]:

$$y = \beta(T - T_0), \quad \rho = r/R, \quad \zeta = z/R, \quad Bi = \alpha R. \tag{5}$$

Then the dimensionless temperature distribution $y(\zeta, \rho)$ results from the following boundary value problem of Fourier's equation with a nonlinearity given by $W = \delta \exp(y)$:

$$\frac{\partial^2 y}{\partial \zeta^2} + \frac{\partial^2 y}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial y}{\partial \rho} + \delta \exp(y) = 0$$

$$\frac{\partial y}{\partial \rho} + (Bi)y = 0 \quad \text{for } \rho = 1, \tag{6}$$

$$y = 0 \quad \text{for } \zeta = \pm L/2R.$$

On account of the symmetry of problem (6), the nonnegative solution y possesses a unique single maximum $y_m = y(0,0)$. From (2), the parameter δ is derived as (cf. [8]):

$$\delta = \delta(\omega) = \tfrac{1}{2} K \sigma_0^2 \omega^{1-n} \beta R^2/\kappa. \tag{7}$$

Here, β, K and n are the parameters appearing in (2) and κ designates the thermal conductivity of the material.

As described in [4], it is possible to consider δ as a dependent parameter by introducing the maximum y_m as an independent parameter, whereby the solutions of problem (6) are obtained in the following parametric representation:

$$(y(\zeta, \rho; y_m), \ \delta(y_m)). \tag{8}$$

From the definition of y_m, one derives that:

$$y(0, 0; y_m) \equiv y_m \tag{9}$$

$\delta(y_m)$ assumes the role of a response function, with the extrema of $\delta(y_m)$ relating to the branching points of problem (6). From the viewpoint of physical applications, the critical points of (6) are of particular interest. Critical points are particular branching points which separate branches of stable solutions from branches of unstable solutions of problem (6). It can be shown that critical points always correspond to an extremum of $\delta(y_m)$. Because of the fact that ($y \equiv 0$, $\delta = 0$) represents a stable solution of (6) which can be analytically continued, it can be deduced that $\delta(y_m)$ contains a branch emerging from $y_m = 0$, $\delta = 0$, such that the solutions (8) of (6) related to this branch of $\delta(y_m)$ are stable solutions. For the particular nonlinearity [$\sim \exp(y)$] of problem (6), it can be shown that only a single critical solution exists (cf. Gray and Lee [2] in their discussion of the result of Steggerda [10], Istratov and Librovich [11]), which then necessarily corresponds to a maximum δ_c of $\delta(y_m)$ in which ends the "stable" branch of $\delta(y_m)$ emerging from $y_m = 0, \delta = 0$. It is the aim of the theory of heat explosions [1] to determine the critical solutions of (6) and, in particular, the critical values δ_c of δ: a maximum δ_c of $\delta(y_m)$, for instance, signifies the occurrence of values $\delta > \delta_c$ for which no solutions of problem (6) exist. The condition of nonexistence of a stationary thermal state [given by either a stable or an unstable solution of problem (6)] is taken to signify the occurrence of a thermal explosion [1].

In [4] it was demonstrated how to get upper and lower bounds for those branches of $\delta(y_m)$ which correspond to stable solutions of (6). It will be demonstrated presently how this method can be carried further to obtain upper and lower bounds of the critical value δ_c of δ.

3. THE CONSTRUCTION OF BOUNDS OF $\delta(y_m)$ BY LINEAR METHODS

For a given solution maximum y_m with $0 < y_m < \infty$,

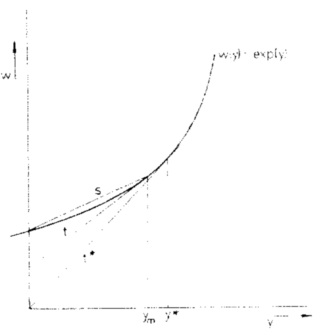

FIG. 1. Linear majorants and linear minorants of $w(y) = \exp(y)$ for a given value of y_m.

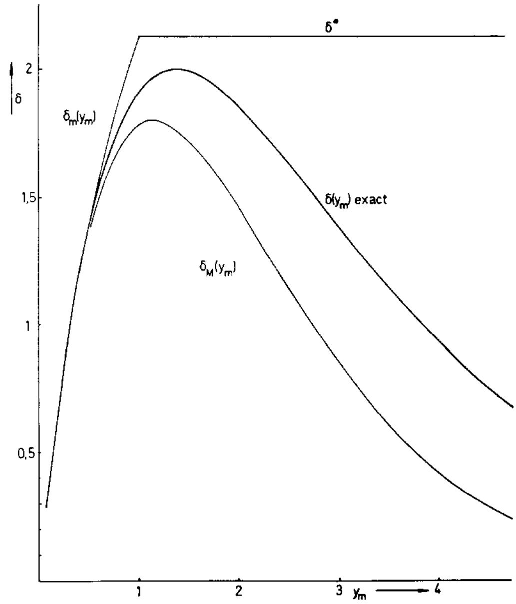

FIG. 2. Continuous upper and lower bounds of $\delta(y_m)$ for an infinite circular cylinder with $Bi = \infty$.

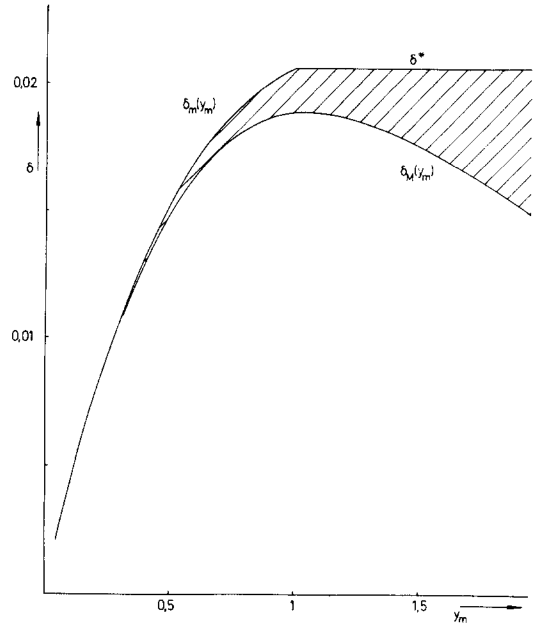

FIG. 4. Continuous upper and lower bounds of $\delta(y_m)$ for a finite cylinder with $Bi = 0.1$.

the closest linear majorant to $w(y) = \exp(y)$ on $0 \leqq y \leqq y_m$ is the secant $s(y, y_m)$ (cf. Fig. 1):

$$s(y, y_m) = (\exp(y_m) - 1)\, y/y_m + 1. \qquad (10)$$

Insertion of (10) for $\exp(y)$ in problem (6) and solving the resulting linear problem under the condition that its solution is to attain the maximum value y_m for $\zeta = 0$, $\rho = 0$ furnishes a value $\delta_M(y_m)$ for which holds that:

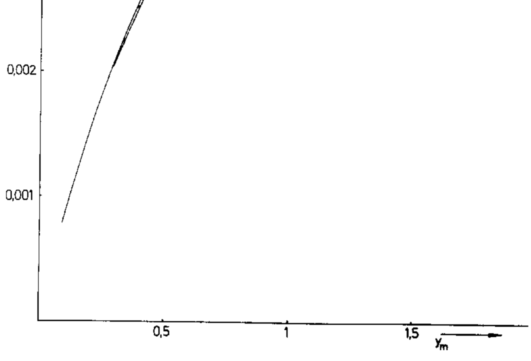

FIG. 3. Continuous upper and lower bounds of $\delta(y_m)$ for a finite cylinder with $Bi = 0$.

$$\delta_M(y_m) \leqq \delta(y_m). \qquad (11)$$

For a given value of y_m with $0 < y_m < 1$, the closest positive linear minorant to $w(y) = \exp(y)$ is the tangent $t(y, y_m)$ (cf. Fig. 1):

$$t(y, y_m) = y \exp(y_m) + (1 - y_m)\exp(y_m). \qquad (12)$$

Insertion of (12) for $\exp(y)$ in problem (6) and solving the resulting linear problem under the condition that its solution is to attain the maximum value y_m for $\zeta = 0$, $\rho = 0$ furnishes a value $\delta_m(y_m)$ for which holds that:

$$\delta(y_m) \leqq \delta_m(y_m). \qquad (13)$$

For $y_m = 1$, a critical tangent t^* is obtained which passes through the origin (cf. Fig. 1). From t^*, a value δ^* results which constitutes a universal upper bound for $\delta(y_m)$ for any value of y_m (cf. Ratner and Koborov [3], Hudjaev [12]).

In order to show how the linear approximation method works and in order to give an impression of how good the bounds are which can be derived, the proposed method is applied to the problem of an infinite circular cylinder where the exact solution $\delta(y_m)$ is known for any value of Biot's number: $(Bi) > 0$ (cf. [6]). In Fig. 2 the result is given for $Bi = \infty$. For $Bi = \infty$, the exact value $\delta_c = 2$ results and one finds the lower bound $\delta_{Mc} = 1.815$ and for the upper bound $\delta_{mc} = 2.1275$.

For $Bi = 0.4$, the exact value $\delta_c = 0.26639$ results and one finds the lower bound $\delta_{Mc} = 0.259$ and the upper bound $\delta_{mc} = 0.2668$.

The same method will now be applied to the problem of a finite cylinder of the length $L = 30$ mm,

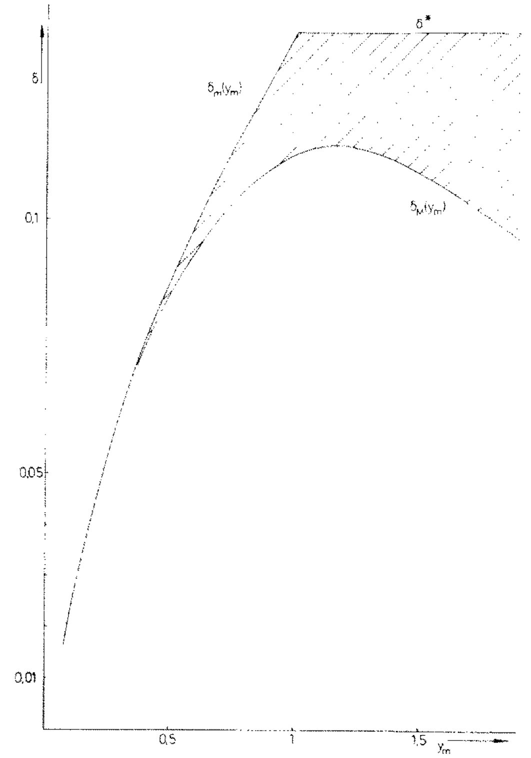

FIG. 5. Continuous upper and lower bounds of $\delta(y_m)$ for a finite cylinder with $Bi = \infty$.

and the radius $R = 4$ mm, which has been under investigation in [4] also. For this sample shape, the exact solution of problem (6) is not known. The resulting bounds from the approximation by linear majorants and minorants for the Biot numbers $Bi = 0$, $Bi = 0.1$ and $Bi = \infty$ are given in Figs 3–5. The upper and lower bounds of the critical values δ_c are listed in Table 1.

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Table 1. Upper bound δ^* and lower bound δ_{Mc} of the critical value δ_c for various Biot-numbers Bi for a finite circular cylinder with $L = 30$ mm and $R = 4$ mm

	$Bi = 0$	$Bi = 0.1$	$Bi = \infty$
δ^*	0.00404	0.02071	0.13701
δ_{Mc}	0.00368	0.0189	0.11471

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THEORIE DE L'EXPLOSION THERMIQUE ET DU CHAUFFAGE VIBRATIONNEL DES POLYMERS

Résumé—Lorsqu'une éprouvette cylindrique de polymère est soumise à une contrainte sinusoïdale d'amplitude constante, des fréquences peuvent apparaître pour lesquelles une 'explosion thermique' conduit à la destruction de l'éprouvette. On montre comment les frontières de ce domaine de fréquences critiques peuvent être déterminées par application des méthodes linéaires.

THEORIE DER WÄRMEEXPLOSION UND DAS AUFHEIZEN VON POLYMEREN DURCH SCHWINGUNGEN

Zusammenfassung—Beim Dauerschwingversuch mit konstanter Spannungsamplitude an einer zylindrischen Polymerprobe können Schwingungsfrequenzen auftreten, bei denen eine Wärmeexplosion zur Zerstörung der Probe führt. Es wird gezeigt, wie die Ober- und Untergrenzen dieser kritischen Frequenzen durch Anwendung linearer Methoden abgeleitet werden können.

ТЕОРИЯ ТЕПЛОВОГО ВЗРЫВА И ВИБРАЦИОННЫЙ НАГРЕВ ПОЛИМЕРОВ

Аннотация—При синусоидальном нагружении цилиндрического полимерного образца с постоянной амплитудой напряжения возникают вибрационные частоты, при которых «тепловой взрыв» приводит к разрушению образца. Показано, каким образом в линейном приближении можно определить верхние и нижние пределы таких частот.